International Journal of Theoretical Physics, Vol. 35, No. 1, 1996

Three-Valued Derived Logics for Classical Phase Spaces

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Received January 23, 1995

Reichenbach proposed a three-valued logic to describe quantum mechanics. In his development, Reichenbach presented three different "negation" operators without providing any criteria for choosing among them. In this paper we develop two three-valued *derived logics for classical systems*. These logics are derived in that they are based on a theory of physical measurement. In this regard they have some of the characteristics of the quantum logic developed by Birkhoff and von Neumann. The theory of measurement used in the present development is the one used previously in developing *bivalent derived logics for classical systems*. As these systems are derived logics, many of the ambiguities possessed by systems such as Reichenbach's are avoided.

1. DERIVED LOGICS

Westmoreland and Schumacher (1993) discussed the notion of "derived logics." The prime motivation for this discussion was to adumbrate the status of von Neumann–Birkoff-type logics which have been proposed for the analysis of quantum mechanical systems. The term "derived logic" reflects the fact that the object language of a physical theory is derived from the mathematical structure of the space of physical states for a given system. For example, the von Neumann–Birkhoff object language of quantum mechanics is derived from the Hilbert space structure that standard quantum mechanics associates with a physical system.

Let us consider a simpler example of a derived logic in order to illustrate the concept. We take our system to be a point particle which is constrained

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0020-7748/96/0100-0031$09.50/0

to lie somewhere along a line. R is the mathematical space of configurations for this system. We operate under the (unphysical) assumption that measurements of the particle's location are infinitely precise. Thus, we will be able to identify measurements with arbitrary subsets of R—for example, infinite precision allows us to determine experimentally whether the particle's coordinate is a rational number or not. A measurement in this dreamland consists in the unambiguous determination of whether the system's configuration lies within a particular set or not.

In this system, propositions about the particle would be of the form: "The particle is in X," where X is a subset of R representing positions. In this case, propositions about the particle will also be identified with subsets of R. This is a result of the fact that any two distinct subsets of R may be distinguished from each other by some measurement. Indeed, given sets A and B in R, the measurement (subset) $A \cap B^c$ (where the superscript c stands for complementation) will distinguish between the sets.

Since we wish for logical operators to map sets of propositions to a proposition, we will identify logical operators with set operations as follows:

$$A \vee B = A \cup B \tag{1}$$

$$A \wedge B = A \cap B \tag{2}$$

$$\neg A = A^c \tag{3}$$

$$A \rightarrow B = A^c \cup B \tag{4}$$

The expression on the right-hand side of each equation treats the objects A and B in their ordinary set-theoretic guises, each expression yielding the definition of the logical operator acting on A and B, now treated as propositions, on the left-hand side.

These definitions accord with our intuition concerning the truth of a proposition as determined by a measurement. For example, if a measurement yields true (respectively, false) for A and true (respectively, false) for B, the same measurement will yield true for $A \cap B$ (respectively, false for $A \cup B$). Similarly, $\neg A$ is determined to be true by a measurement (respectively, false) if and only if A is determined to be false (respectively, true) by the same measurement.

Given this mathematical state space and this theory of measurement, the derived logic for this system will be ordinary Boolean logic. This follows from the fact that the calculus of set operations is a Boolean lattice. It is interesting to note that a theory of measurement which is only slightly more sophisticated than the one considered here will yield non-Boolean logics.

The main thesis of Westmoreland and Schumacher (1993) is that nonstandard derived logics are possible for a wider class of physical theories than quantum mechanics. To illustrate this fact, two nonstandard logics are derived for classical mechanical systems. The nonstandard features of von Neumann–Birkhoff logic flow from the Hilbert space (more specifically, the vector space) structure of quantum mechanics. The nonstandard features of the phase space logics derived in Westmoreland and Schumacher (1993) flow from the topological structure of the phase space associated with a physical system by classical dynamics. More specifically, the open phase space logic is derived from the structure of open subsets of the phase space, while closed phase space logic is derived from the structure of closed subsets of the phase space. These logics are motivated by a rudimentary theory of measurement, in which a given measurement outcome localizes the state of the system to an open set in phase space rather than an idealized point. Subsets of phase space which cannot be distinguished (in one sense or another) by such measurements must be identified with the same proposition in the derived logic.[4]

As noted in Westmoreland and Schumacher (1993), the law of *tertium non datur* does not hold in open phase space logic. The possibility that the law of *tertium non datur* is not a tautology has previously been a motivation for the development of three-valued logics as possible alternatives to standard Boolean logic (Łukasiewcz, 1970; Reichenbach, 1944; Rosser, 1941; Lewis and Langford, 1932; Rosser and Turquette, 1952). Thus, the possibility that a three-valued logic for classical mechanical systems is suggested. It is the purpose of this paper to develop and discuss two possible three-valued logics for classical systems as derived logics. One reason for developing a three valued *derived* logic is that we may hope to thereby avoid the ambiguity which often attends the selection of three-valued connectives. As pointed out in Rosser (1941), there are 12 distinct "negations" which are possible for a three-valued logic. The choice among these is to some degree ad hoc. Indeed, in the three-valued system he developed for quantum mechanics, Reichenbach (1944) chose to include three different negations. As we will show, the derivation of a three-valued logic from the mathematical structure of the state space eliminates some of the ad hoc features of the choice of connectives.

2. OPEN PHASE SPACE LOGIC

Westmoreland and Schumacher described two derived logics for classical mechanical systems; the open phase space logic and the closed phase space

[4] These derived phase space logics are distinct from the open set "logic of affirmative assertions" discussed by Vickers (1989), though they share a common motivation. Both approaches recognize that any set of measurements must be finite and thus can only provide approximate knowledge of the state of a system. However, Vickers identifies the propositions with open sets in the state space, rather than equivalence classes of sets that are indistinguishable by open sets. This leads to several differences; for example, the "logic of affirmative assertions" lacks a negation operator.

logic. As *tertium non datur* does not hold in open phase space logic (see Theorem 4 below), it is a natural candidate for a three-valued interpretation. In this section we describe those properties of the open phase space logic which deal with the structure of the propositions in the logic. As these are the same for both the bivalent and three-valued versions of open phase space logic, we will not distinguish between the two in the present section. In Section 3 we will exhibit statements about propositions in the three-valued logic and indicate how these differ from the bivalent case.

Let P be a proposition; we recall from Westmoreland and Schumacher (1993) that this means that P is an equivalence class of sets. Distinct sets are in the same equivalence class P if and only if they have the same interior, as sets with the same interior cannot be verified by the same set of measurements (which are to be identified with open sets, as noted in the previous section). The properties of these propositions are determined by the topological properties of subsets of the phase space. Suppose V and W are arbitrary subsets of a topological space X. Then it is true that

$$\overline{\overline{V}} = \overline{V} \tag{5}$$

$$\operatorname{int}(\operatorname{int} V) = \operatorname{int} V \tag{6}$$

$$\overline{V} \cap \overline{W} \supseteq \overline{V \cap W} \tag{7}$$

$$\overline{V} \cup \overline{W} = \overline{V \cup W} \tag{8}$$

$$\operatorname{int} V \cup \operatorname{int} W \subseteq \operatorname{int}(V \cup W) \tag{9}$$

$$\operatorname{int} V \cap \operatorname{int} W = \operatorname{int}(V \cap W) \tag{10}$$

($\overline{V}$ and int V denote the closure and interior of the set V, respectively.) Proofs of the above facts can be found in any introductory topology text (e.g., Munkres, 1974).

As we stated earlier, we wish to find a single alternative proposition to P; this single alternative proposition will serve as the negation of P. The alternative which is readily suggested is $[\operatorname{int}(P_i^c)]$ where P_i is any element of P. This choice of the alternative proposition motivates our definitions of the three-valued operations corresponding to "or," "and," and "not." These operators are defined as follows:

Definition 2.1. Let P and Q be propositions. That is, P is the equivalence class $[P_i]$ and Q is the equivalence class $[Q_j]$. Then we define the logical operators $\neg$ ("not"), $\wedge$ ("and"), and $\vee$ ("or") as follows:

$$\neg P = [(\operatorname{int} P_i)^c] \tag{11}$$

$$P \wedge Q = [(\operatorname{int} P_i) \cap (\operatorname{int} Q_j)] \tag{12}$$

$$P \vee Q = [(\text{int } P_i) \cup (\text{int } Q_j)] \tag{13}$$

These definitions are all based on the notion that a proposition is to be identified with a substructure of the mathematical state space, in this case, an equivalence class of sets.

Theorem 2.2. The operations $\vee$ and $\wedge$ are both commutative and associative. Furthermore, $\vee$ distributes over $\wedge$, and $\wedge$ distributes over $\vee$.

Proof. We will omit the proofs of commutativity and associativity, and give only the proof for the distributivity of $\wedge$ over $\vee$. Other parts of the proof are straightforward.

Suppose A, B, and C are propositions in open phase space logic. Then

$$\begin{aligned} A \wedge (B \vee C) &= [\text{int } A_\mu] \wedge [\text{int } B_\nu \cup \text{int } C_\eta] \\ &= [\text{int } A_\mu \cap \text{int}(\text{int } B_\nu \cup \text{int } C_\eta)] \end{aligned}$$

Referring to the topological properties listed in (1)–(6), we see that

$$\begin{aligned} A \wedge (B \vee C) &= [\text{int } A_\mu \cap (\text{int } B_\nu \cup \text{int } C_\eta)] \\ &= [(\text{int } A_\mu \cap \text{int } B_\nu) \cup (\text{int } A_\mu \cap \text{int } C_\eta)] \end{aligned}$$

The intersection of two open sets is open, so

$$\text{int } V \cap \text{int } W = \text{int}(\text{int } V \cap \text{int } W) \tag{14}$$

Thus,

$$\begin{aligned} A \wedge (B \vee C) &= [\text{int}((\text{int } A_\mu \cap \text{int } B_\nu)) \cup \text{int}((\text{int } A_\mu \cap \text{int } C_\eta)) \\ &= [\text{int } A_\mu \cap \text{int } B_\nu] \vee [\text{int } A_\mu \cap \text{int } C_\eta] \\ &= (A \wedge B) \vee (A \wedge C) \end{aligned}$$

Therefore, $\wedge$ distributes over $\vee$ in open phase space logic. The proof that $\vee$ distributes over $\wedge$ is exactly similar. ∎

Theorem 2.3. For a proposition P, $P \sqsubseteq \neg\neg P$.

Remark. Recall from Westmoreland and Schumacher (1993) that $P \sqsubseteq Q$ means that for every set P_i in the equivalence class P there is a set Q_j in Q such that $P_i \subseteq Q_j$.

Proof. Let P be a proposition; i.e.;

$$\begin{aligned} P &= [P_i] \\ &= [\text{int } P_i] \end{aligned}$$

For any set S it is the case that int $S \subseteq$ int((int $(S^c))^c$). This can be seen as follows: Let $x \in$ int S. Then there is a neighborhood of x, denoted by U_x, such that $U_x \subseteq$ int $S \subseteq S$. Hence, $U_x \cap S^c = \emptyset$ and so $U_x \cap \text{int}(S^c) = \emptyset$. Thus $U_x \subseteq (\text{int}(S^c))^c$, so that $x \in \text{int}((\text{int}(S^c))^c)$ and we have int $S \subseteq \text{int}((\text{int}(S^c))^c)$ as claimed. Thus, we may continue the proof of the theorem:

$$\begin{aligned} P &\sqsubseteq [\text{int}((\text{int}(P_i^c))^c)] \\ &= \neg[(\text{int}(P_i^c))] \\ &= \neg\neg[P_i] \\ &= \neg\neg P \quad \blacksquare \end{aligned}$$

Theorem 2.4. In open phase space logic, $A \wedge \neg A = 0$ ("0" here denotes the equivalence class which contains the empty set, which is the never-verified, or always-false, proposition) for all propositions A (law of noncontradiction). On the other hand, $A \vee \neg A \neq 1$ for some topological spaces; i.e., no *tertium non datur* ("1" here denotes the equivalence class which contains the entire space, which is the always-verified, or always-true, proposition).

Proof. By the definitions of $\vee$ and $\neg$ we have

$$A \wedge \neg A = [\text{int } A_\mu] \wedge [(\text{int } A_\nu)^c]$$

where A_μ and A_ν are any representatives of A. Let us choose both to be the canonical open representative A_ω. Then

$$A \wedge \neg A = [A_\omega \cap \text{int}((A_\omega)^c)]$$

Since $V \cap V^c = \emptyset$ for any subset V of X, and since int $V \subseteq V$, it follows that $A_\omega \cap \text{int}((A_\omega)^c) = \emptyset$. Thus, $A \wedge \neg A = 0$, and so the law of noncontradiction holds.

On the other hand, suppose that our phase space is R^2 and that our proposition A is the equivalence class of sets whose complement is the y axis. Then $\neg A = 0$, so that $A \vee \neg A = A \vee 0 = A \neq 1$, and so *tertium non datur* does not hold. $\blacksquare$

Theorem 2.5. For Propositions A and B in the open phase space logic,

$$\neg(A \vee B) = \neg A \wedge \neg B \tag{15}$$

$$\neg A \vee \neg B \sqsubseteq \neg(A \wedge B) \tag{16}$$

In the second relation, equality need not hold.

The proofs of these facts are straightforward. The two relations work out differently because of the asymmetry between set union and intersection with respect to the interior operation. That is, by the topological properties

(1)–(6), the interior of the intersection of two sets is equal to the intersection of their interiors; however, in general the union of their interiors merely contains the interior of the union of those sets.

It is interesting to note that the open phase space logic is Boolean if the underlying phase space has a discrete topology. (The phase spaces of ordinary classical dynamical systems, of course, do not.) Thus, the topological structure of the state space governs the character of the derived logic.

We reiterate that these properties are independent of whether the statements are interpreted using a bivalent (P is verified or not verified) or a three-valued interpretation. For a discussion of the bivalent interpretation of open phase space logic see Westmoreland and Schumacher (1993). The description of the three-valued interpretation follows in Section 3.

3. THREE-VALUED OPEN PHASE SPACE LOGIC

In the rudimentary theory of measurement described by Westmoreland and Schumacher (1993), measurements were identified with open subsets of the phase space. This was motivated by the fact that classical measurements are inherently imprecise. With this in mind, it is said that a measurement (open set) m verifies a proposition P if and only if $m \subseteq P_i$ for every set $P_i \in P$. These are the meanings we assign to the words "measurement" and "verify" throughout the remainder of this paper.

Using a bivalent metalanguage in open phase space logic, we speak of verification as opposed to truth and falsehood. That is, a proposition P is verified or it is not verified. One might be tempted to call those propositions that are verified "true" and those that are not verified "false." The following description will show why this is not satisfactory.

Let S be a classical system. Let P be a proposition about S in open phase space logic with P_0 as the canonical representative of $[P]$. That is, $P_0 = \text{int}(P_j)$ for all $P_j \in [P]$. A measurement m_1 that contains points of P_0 but does not lie entirely in P_0 would not verify P. If we used truth and falsehood instead, we would say that P is false. It is clear that the measurement m_1 provides us with an ambiguous result. That it does not verify P is clear; it does not lie entirely within P_0. Does this mean that we should conclude that P is false? The state of S could lie in P_0 and still be consistent with the result of the measurement m_1. Indeed, according to our naive measurement theory, there is a more precise measurement m which lies entirely in P_0 and the result of m_1. Thus, we see that we cannot conclude that P is false.

As further evidence, let us now consider a measurement result m_2 that is contained entirely in $[\text{int}(P_0^c)]$; i.e., m_2 contains no points of P_0. Again, it is clear that this new measurement does not verify P. If we insist on using "truth" and "falsehood" instead of "verification," does the measurement m_2

attribute the same degree of falsehood to P as m_1 does? There is no more precise measurement of S consistent with m_2 that can verify P. In fact, the equivalence class containing the complement of P_0 is verified. There exist sets in the equivalence class P that have points in the result of m_2, but none of these are interior points of P_0. Thus, according to our measurement theory, we cannot restrict the state S to these sets. Consequently, the measurement m_2 is consistent with P being false.

The argument above shows why it is troublesome to use the values "true" and "false" when using a bivalent language for open phase space logic. This is why the bivalent interpretation of open phase space logic is phrased strictly in terms of verifiability. One means of incorporating "truth" and "falsehood" into open phase space logic is to use a three-valued interpretation of the logic. Let us now consider the possibility of implementing this approach.

As we have seen in the discussion above, when a proposition P is not verified, it does not follow that the negation of P is verified (i.e., "P being not verified" is not equivalent to "$\neg P$ being verified"). Similarly, if the negation of P is not verified, we cannot conclude that P is verified. In other words, it is possible that a proposition and its negation may both fail to be verified. This is a reflection of the fact that *tertium non datur* does not hold in open phase space logic; i.e., $P \vee \neg P \neq 1$. As the canonical representatives of P and $\neg P$ do not cover the entire space, one can reasonably expect there to be another alternative. It is the existence of this third alternative that indicates that a three-valued interpretation may be useful for open phase space logic.

There are two points worth noting at this juncture: First, when dealing with a three-valued logic one must take some care with defining the negation of a proposition. This contrasts with the situation one finds in a two-valued logic. In the two-valued case, only one negation is possible, but in a three-valued logic, the existence of more than one alternative value for a given proposition implies that more than one negation is possible. Indeed, in the case of the three-valued logic which Reichenbach developed for quantum mechanics, he identified three distinct negations. In the present development, as our analysis uses a derived logic, we are able to choose one negation; as we demonstrated in Section 2, this choice of one negation is possible because, in a derived logic, logical connectives are associated with substructures of the mathematical state space which models the physical system.

The following definitions will aid in establishing a three-valued interpretation for open phase space logic.

Definition 3.1. For a set S in the topological space X, we define the *boundary* of S to be the set $\partial S = \overline{S} \cap \overline{S^c}$. A point x is a *boundary point* of S if $x \in \partial S$.

Note: This definition implies that, for a given set, the interior and the boundary of that set are disjoint. Also, for a given set A, it is the case that $\bar{A} = \text{int}(A) \cup \partial(A)$.

Definition 3.2. Let S be a set in the topological space X. Then a point y on the boundary of S is an *adherent point* iff every neighborhood of y contains a point of the interior of S. The set of all the adherent points of S is denoted by *adhS*.

A few words may be in order concerning the idea of adherent point. Consider a set P_i as an element of the proposition P. Roughly speaking, the adherent points of P_i are those boundary points which cannot be "pried away from P_i" by any measurement. That is to say, $x \in \partial P_i$ is an adherent point of P_i if, for any open set m, (such as those representing measurements) containing x is such that $m \cap P_i$ has a nonempty interior. Stated yet another way: $x \in \partial P_i$ is an adherent point if any measurement which contains x must also contain some of the interior of P_i.

We want to look at some results involving adherent points, as this category of points will prove useful in describing the three-valued open phase space logic. First, we will prove a result involving the interiors and complements of sets.

Theorem 3.3. For a set S in a topological space X, $(\text{int}(S))^c = \overline{S^c}$.

Proof. Let $x \in (\text{int}\, S)^c$. We know that $\text{int}\, S \subset \bar{S} = \text{int}\, S \cup \partial S$. As $x \notin \text{int}\, S$, this implies that $x \in \partial S$ or $x \in (\bar{S})^c$. If $x \in \partial S$, then $x \in \overline{S^c}$ by Definition 1. On the other hand, if $x \in (\bar{S})^c$, then there exists a neighborhood U_x of x that does not intersect $\bar{S}$ [this follows from the fact that $(\bar{S})^c$ is an open set]. Thus $U_x \cap S = \emptyset$, so $x \in U_x \subset S^c \subset \overline{S^c}$. In either case ($x \in \partial S$ or $x \in \overline{S^c}$) we have that $x \in \overline{S^c}$, so we may conclude that $(\text{int}(S))^c \subset \overline{S^c}$.

For the reverse containment, let $x \in \overline{S^c}$. By the definition of closure, every neighborhood of x intersects S^c. As $\text{int}\, S \subset S$, we have that $S^c \subset (\text{int}\, S)^c$, so every neighborhood of x also intersects $(\text{int}\, S)^c$. That is, $x \in \overline{(\text{int}\, S)^c}$. As int S is an open set, we know that $(\text{int}\, S)^c$ is its own closure. Hence, $x \in (\text{int}\, S)^c$, which implies that $\overline{S^c} \subseteq (\text{int}\, S)^c$. As we have containment in both directions, we have proved that $\overline{S^c} = (\text{int}\, S)^c$. ∎

We now turn to several theorems concerning adherent points, as promised.

Theorem 3.4. If S is an open set, then the set of boundary points of S is equal to the set of adherent points of S; i.e., $\partial S = adh(S)$.

Proof. Let $x \in \partial S$; this implies that $x \in \bar{S}$. Thus every neighborhood of x contains a point $y \in S$. As S is an open set, we have that $y \in \text{int}(S)$.

Hence every neighborhood of x contains a point in the interior of S, so $x \in adh(S)$. Thus, $\partial S \subset adh(S)$. By definition we know that $adh(S) \subset \partial S$, so we have that $adh(S) = \partial S$. ■

Theorem 3.5. Let $[R] = [S]$ [i.e., let $\text{int}(R) = \text{int}(S)$]; then $adh(R) = adh(S)$.

Proof. Let $x \in adh(R)$. By the definition of adherent point, we know that $x \in \partial R$ and that every neighborhood of x intersects $\text{int}(R)$. As $\text{int}(R) = \text{int}(S)$ by assumption, we then have that every neighborhood of x intersects $\text{int}(S)$. Thus, in order to establish that $x \in adh(S)$ we need to show that $x \in \partial S$. As $x \in \partial R = \overline{R} \cap \overline{R^c}$ we have that $x \in \overline{R^c}$. By Theorem 3.5 we have that $\overline{R^c} = (\text{int}(R))^c = (\text{int}(S))^c = \overline{S^c}$, so that $x \in \overline{S^c}$. As every neighborhood of x intersects $\text{int}(S)$ we see that we also have that $x \in \overline{S}$, so that $x \in \overline{S} \cap \overline{S^c} = \partial S$. Thus we have established that $x \in adhS$; consequently, $adh(R) \subset adh(S)$. As the reverse containment follows by the symmetry in our assumption on R and S, the desired equality follows. ■

Remark. It should be noted that Theorem 3.5 allows us to talk of the *set* of adherent points associated with the *proposition* P, even though P is an equivalence class of sets. Hence we may unambiguously speak of $adh(P)$ even when P is a proposition and not just a set of points.

The following topological fact will be useful when we compare a proposition P to its double negation $\neg\neg P$.

Theorem 3.6. For any proposition P, $adh(\neg\neg P) \subseteq adh(P)$.

Proof. Let P_0 be the canonical representative of P and let $Q_0 = \text{int}\{[\text{int}(P_0^c)]^c\}$ be the canonical representative of $\neg\neg P$ (as we saw in Theorem 2.3 that $P \neq \neg\neg P$, so there is no reason to expect that $P_0 = Q_0$). Theorems 3.4 and 3.5 imply that the present result will be established if we prove that $\partial(Q_0) \subseteq \partial(P_0)$.

Let $x \in \partial(Q_0)$; then by the definition of Q_0 and the definition of the boundary of a set we have that

$$x \in \overline{\text{int}([\text{int}(P_0^c)]^c)} \cap \overline{\{\text{int}([\text{int}(P_0^c)]^c)\}^c}$$

We must first establish two facts about the sets in this intersection:

1. $\overline{\{\text{int}([\text{int}(P_0^c)]^c)\}^c} \subseteq P_0^c$.
2. $\overline{\text{int}([\text{int}(P_0^c)]^c)} \subseteq \overline{P_0}$.

The derivation of item 1 is as follows:

$$\text{int}(P_0^c) \subseteq P_0^c$$

$$[\text{int}(P_0^c)]^c \supseteq (P_0^c)^c$$

$$\begin{aligned}
\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\} &\supseteq \mathrm{int}\{(P_0^c)^c\} \\
&= \mathrm{int}(P_0) \\
&= P_0 \\
(\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\})^c &\subseteq P_0^c \\
\overline{(\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\})^c} &\subseteq \overline{P_0^c}
\end{aligned}$$

Item 2 is derived similarly:

$$\begin{aligned}
P_0 &\subseteq \overline{P_0} \\
P_0^c &\supseteq (\overline{P_0})^c \\
\mathrm{int}(P_0^c) &\supseteq \mathrm{int}\{(\overline{P_0})^c\} \\
&= (\overline{P_0})^c \\
[\mathrm{int}(P_0^c)]^c &\subseteq \{(\overline{P_0})^c\}^c = \overline{P_0} \\
\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\} &\subseteq \mathrm{int}(\overline{P_0}) \subseteq \overline{P_0} \\
\overline{\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\}} &\subseteq \overline{\overline{P_0}} = \overline{P_0}
\end{aligned}$$

We are now able to prove that $\partial Q_0 \subseteq \partial P_0$ as follows:

$$\begin{aligned}
x &\in \overline{\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\}} \cap \overline{\{\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\}\}^c} \\
&\subseteq \overline{\mathrm{int}\{[\mathrm{int}(P_0^c)]^c\}} \cap \overline{P_0^c} \\
&\subseteq \overline{P_0} \cap \overline{P_0^c} \\
&= \partial P_0
\end{aligned}$$

Thus $\partial Q_0 \subseteq \partial P_0$, which establishes our result. ■

For a proposition P in bivalent open phase space logic, we saw that a measurement might verify P or it might not verify P. However, if a measurement does not verify P, we cannot conclude that this measurement gives us that P is false. We will now describe a three-valued open phase space logic in which propositions *may* be naturally assigned values of "true" or "false" by a given measurement. The price to be paid is that a given measurement may also result in a value of "indeterminate" being assigned to P. We shall assign truth values according to the following criteria:

Definition 3.7. Let P be a proposition in open phase space logic and let m be a measurement result. Suppose that P_i is any set in $[P]$. Then the proposition P will be assigned the truth value:

(1) **true** if m verifies P;
(2) **false** if m does not verify P but $m \cap adh(P_i) = \emptyset$;
(3) **indeterminate** if m does not verify P, but $m \cap adh(P_i) \neq \emptyset$.

A few observations concerning this definition are in order. In view of our measurement theory, these three truth values are sufficient for all possible measurements of a system designed to verify a proposition P. Theorem 3.5 implies that all sets in a given proposition (an equivalence class) will have the same set of adherent points. Hence the assignment provided by Definition 3.7 is well defined. As measurements are identified with open sets, any measurement of P which contains adherent points of a set $P_i \in P$ must also contain interior points of P_i. Thus, there exists a more precise measurement consistent with the current measurement which might verify P; i.e., the measurements which contain adherent points of the representatives of P are precisely those measurements which do not verify P and were inconclusive as to whether or not P is false.

We now turn our attention to statements whose interpretation in bivalent open phase space logic differs from its interpretation in three-valued open phase space logics. Recall that in bivalent open phase space logic our "truth values" were "verified" and "unverified." One of our motivations for the three-valued logic developed here was to recover a natural notion for the valuation of propositions as being "true" or "false." Of course the notions of verified and truth are related: We say, using the three-valued logic, that a measurement determines that a proposition is true if the measurement verifies the proposition and a proposition is determined to be false if the measurement verifies the negation of the proposition. As we have seen, we cannot equate the ideas of "true" and "verified," as it is conceivable that neither a proposition nor its negation will be verified by a suite of measurements. This is the reason why we need the value of "indeterminate" along with those of "true" and "false."

Theorem 3.10 below is an example of a statement which can be made using three-valued open phase space logic which has no natural analog in the bivalent open phase space logic. We may think of this theorem as a weaker form of *tertium non datur.* Before we turn to the proof of Theorem 3.10 we need to establish two technical lemmas concerning adherent points associated with an element of a proposition of the form $P \vee \neg P$.

Lemma 3.8. If Q is an open set, then $Q \cup \mathrm{int}(Q^c) = [adh(Q)]^c$.

Proof. Let $x \in Q \cup (\text{int}(Q^c))$; then $x \in Q$ or $x \in \text{int}(Q^c)$. Consider the case where $x \in Q$. As Q is an open set, we know by Theorem 3.4 that $\partial Q = adh(Q)$. Also, as Q is open, it contains none of its boundary points, so $x \notin \partial Q = adh(Q)$. So $x \in [adh(Q)]^c$. If $x \in \text{int}(Q^c)$, then there is a neighborhood N_x of x such that $N_x \subset \text{int}(Q^c)$, so $N_x \subset (Q^c)$, which implies that $N_x \cap Q = \emptyset$. Thus $x \notin \partial Q = adh(Q)$. Thus we also have in this case that $x \in [adh(Q)]^c$. Thus $Q \cup \text{int}(Q^c) \subseteq [adh(Q)]^c$. Conversely, let $x \in [adh(Q)]^c$. Then x is not an adherent point of Q, so x is not in the boundary of Q. Thus $x \in \text{int } Q = Q$ or $x \in \text{int}(Q^c)$, i.e., $x \in Q \cup \text{int}(Q^c)$. Thus, we have established that $x \in [adh(Q)]^c \subset Q \cup (\text{int}(Q^c))$ and the equality follows. ■

Lemma 3.9. If Q is an open set, then $adh(Q) = adh(Q \cup (\text{int}(Q^c)))$.

Proof. As $Q \cup (\text{int}(Q^c))$ is the union of open sets, it is an open set, so Theorem 3.4 gives us that $adh(Q \cup (\text{int}(Q^c)) = \partial(Q \cup (\text{int}(Q^c))$. By Lemma 3.8, the boundary of $Q \cup (\text{int}(Q^c))$ is equal to $\overline{[adh(Q)]^c} \cap \overline{([adh(Q)]^c)^c}$. We also know by Lemma 3.8 and the definition of boundary that $\overline{[adh(Q)]^c} = X$ and that $\overline{([adh(Q)]^c)^c} = \overline{adh(Q)} = adh(Q)$. Hence

$$adh(Q \cup (\text{int}(Q^c)) = X \cap [adh(Q)] = adh(Q) \quad ■$$

Theorem 3.10. The proposition $P \vee \neg P$ is never false (i.e., no measurement can verify the negation of $P \vee \neg P$).

Proof. Let P be a proposition. By definition we have

$$\begin{aligned} P \vee \neg P &= [P_i] \vee [(\text{int } P_j)^c] \\ &= [\text{int } P_i] \vee [\text{int}((\text{int } P_j)^c)] \\ &= [\text{int } P_i \cup \text{int}((\text{int } P_j)^c)] \\ &= [\text{int } P_i \cup \text{int}((\text{int } P_i)^c)] \end{aligned}$$

By Lemma 3.10 we have

$$P \vee \neg P = [(adh(\text{int } P_i))^c] \tag{17}$$

If we let P_0 be the canonical representative of P (i.e., the unique open set in P), then $adh(\text{int } P_i) = \partial P_0$, which is a closed set. Thus we have

$$P \vee \neg P = [(\partial P_0)^c] \tag{18}$$

As ∂P_0 is closed, $(\partial P_0)^c$ is an open set and so it is the canonical representative of $P \vee \neg P$.

If we assume that $P \vee \neg P$ is false, then, by definition, it must be the case that $\neg(P \vee \neg P)$ is verified, but

$$\begin{aligned}\neg(P \vee \neg P) &= \neg[(\partial P_0)^c] \\ &= [\mathrm{int}((\partial P_0)^c)^c] \\ &= [\mathrm{int}(\partial P_0)]\end{aligned}$$

For any set S it is the case that $\mathrm{int}(\partial S) = \emptyset$ (Munkres, 1974). Thus $\neg(P \vee \neg P) = [\emptyset] = 0$. That is, for any proposition P, the negation of $P \vee \neg P$ is always unverified. Hence, its negation can never be verified, so $P \vee \neg P$ is never false. ■

Remark. One might be tempted to prove that $P \vee \neg P$ is never false by involving the equivalence $P \vee \neg P = \neg(P \wedge \neg P)$ and noting that $P \wedge \neg P$ is always false (never verifiable). While it is true that $P \wedge \neg P$ is always false in three-valued open phase space logic (see Theorem 2.4), the proposed equivalence ($P \vee \neg P = \neg(P \wedge \neg P)$) does not hold in open phase space logic (see Theorem 2.5). Consequently, a more involved proof for Theorem 3.10 is called for in three-valued open phase space logic.

The question arises: What can be said about $\neg P$ and $\neg\neg P$ given that a measurement assigns a particular truth value to P? Theorems 3.11–3.13 and Corollary 3.14 provide the answer.

Theorem 3.11. If a measurement assigns true to P, then the same measurement assigns false to $\neg P$ and true to $\neg\neg P$.

Proof. Let P be proposition which is verified by a measurement m. By definition, this means that m is an open set such that $m \subseteq \mathrm{int}(P_i)$, where P_i is any set in P. In order to show that m makes $\neg P$ false, we must show that $m \cap \mathrm{int}(Q_j) = \emptyset$, where Q_j is any element of $\neg P$ (see the discussion following Definition 3.7). By the definition of negation we may set $Q_j = \mathrm{int}(P_i)^c$, where P_i is the element of P described above. As $m \subseteq P_i$ it must be the case that $m \cap (P_i)^c = \emptyset$. Thus, as $\mathrm{int}((P_i)^c) \subseteq (P_i)^c$, we have that $m \cap \mathrm{int}((P_i)^c) = \emptyset$. As m does not verify $\neg P$ and m does not contain any adherent points of $\neg P$, we say that m assigns false to $\neg P$.

In order to prove the second part of the theorem we must show that $m \subseteq \mathrm{int}\, R_k$, where R_k is some element of $\neg\neg P$. By Theorem 3.10 we have that $P \sqsubseteq \neg\neg P$, by which we mean that any element of P is a subset of some element of $\neg\neg P$. Thus for the set P_i described above, there is some $R_i \in \neg\neg P$ such that $P_i \subseteq R_i$; consequently, $m \subseteq \mathrm{int}\, R_i$. Thus m assigns true to $\neg\neg P$. ■

The proof of the following theorem is dual to the proof of Theorem 3.11 in the sense that arguments for $m \subseteq \mathrm{int}\, S$ (where S is a set) are replaced by arguments for $m \cap \mathrm{int}\, S = \emptyset$ and vice versa.

Theorem 3.12. If a measurement assigns false to P, then the same measurement assigns true to $\neg P$ and false to $\neg\neg P$.

We now look at the remaining case: m assigns the value of indeterminate to P.

Theorem 3.13. If a measurement assigns indeterminate to P, then the same measurement may assign true or m may assign indeterminate to $\neg\neg P$.

Proof. We will prove that m will not assign a value of false to $\neg\neg P$ and we then show by example that each of the remaining possibilities is obtained. Let m be a measurement which assigns indeterminate to P. By definition m must contain adherent points of P. By Theorem 3.6, $adh(\neg\neg P)) \subseteq adh(P)$, but by Theorem 3.10, $P \sqsubseteq \neg\neg P$. Thus, if m does not contain any adherent points of $\neg\neg P$, then m must be contained entirely in int R_k, where $R_k \in \neg\neg P$. That is, if m does not assign indeterminate to $\neg\neg P$, then m must assign true to $\neg\neg P$. Thus m cannot assign false to $\neg\neg P$.

For the example of an m assigning indeterminate to P and true to $\neg\neg P$, consider the proposition used in our proof of Theorem 2.4: Suppose that our phase space is R^2 and that our proposition A is the equivalence class of sets whose complement is the y axis. Further, let m be the measurement $m = \{(x, y) | -1 < x < 1\}$. We see that m contains adherent points of A and so m assigns indeterminate to A, but $\neg\neg A = 1$, so m assigns a value of true to $\neg\neg A$.

For the example of an m assigning indeterminate to P and indeterminate to $\neg\neg P$, consider the following proposition: Suppose that our phase space is R^2 and that our proposition B is the equivalence class of sets whose interior is the left half-plane; i.e., the canonical representative is $B_0 = \{(x, y) | x < 0\}$. Again, let m be the measurement $m = \{(x, y) | -1 < x < 1\}$. We see that m contains adherent points of B and so m assigns indeterminate to B and in this case $\neg\neg B = B$, so m also assigns indeterminate to $\neg\neg B$. ∎

Corollary 3.14. If a measurement assigns indeterminate to P, then the same measurement may assign false or m may assign indeterminate to $\neg P$.

Proof. If m is assigned true to $\neg P$, then by Theorem 11, m would assign false to $\neg\neg P$, which contradicts Theorem 3.13. If m only assigned false to $\neg P$, then by Theorem 3.12, m must assign true to $\neg\neg P$, which we have shown to not be the case in general. ∎

This last set of results (Theorems 3.11–3.13 and Corollary 3.14) points up an important fact about the operators in three-valued open phase space logic: the operators are not truth functional. That is to say, a given truth value for a measurement–proposition pair (m, P) does not give a well-defined truth value for an operator acting on P under m. Corollary 3.14 provides a prime example of this lack of truth-functional behavior: If P is assigned the value indeterminate by m, then $\neg P$ may be assigned indeterminate or false by m.

This lack of truth functionality is shared by other derived logics: the bivalent open and closed phase space logics, and the von Neumann–Birkhoff logic for quantum mechanics are all examples of non-truth-functional logics. For a discussion of this property for the von Neumann–Birkhoff logic see Gibbins (1987). That the bivalent phase space logics are not truth functional is closely related to the fact that the three-valued variety of open phase space logic is not truth functional. As we noted above, it is possible for a proposition and its negation to both be unverified. Of course it is also possible that a proposition may be unverified and for its negation to be verified. Thus the negation operator in open phase space logic under the bivalent interpretation is not truth functional.

The fact that these logics are not truth functional is closely related to a perhaps more obvious property which they share: they are not truth-valued. That is, the connectives in these logics do not assign truth values to propositions, rather they assign other propositions to propositions. This contrasts with the situation found in classical logic, where connectives assign truth values to sets of propositions. Indeed, the situation in classical logic is that we may think of operators as being either truth-valued or proposition-valued. Assigning a proposition to an operation acting on other propositions is equivalent to assigning a truth value to the operation based only on the truth values of the propositions upon which the operation is acting. With the derived logics listed above these two assignments are no longer equivalent.

It is interesting to note that the logic derived using the unphysical measurement theory (where the imprecision of measurement is ignored entirely), which we discussed briefly in Section 1, provides an example of a derived logic which is truth functional. Upon reflection, it is seen that this follows from the fact that the space of states is simply the lattice of subsets of the state space which is a Boolean lattice and so is truth functional. One might conjecture that any derived logic under a bivalent interpretation is, in some sense, "isomorphic" to this derived logic. Of course any attempt to address this conjecture must largely be about making precise this notion of "isomorphic derived logics."

4. A THREE-VALUED LOGIC WITH A TRUTH-FUNCTIONAL NEGATION

In Section 3 we described a derived logic in which the negation operator is not truth functional. We observed that having operators which are not truth functional is a property exhibited by many derived logics; indeed, the only bivalent derived logic which is known to be truth functional is the logic associated with the unphysical measurement theory. In this section we will describe a three-valued phase space logic in which the negation operator is truth functional. We will also discuss this logic's status as a *derived* logic.

Recall that the obstacle to the negation operator being truth functional was that a measurement m might assign the value of indeterminate to a proposition P with the possibility that m could assign either a value of indeterminate or false to $\neg P$. Upon closer consideration of the example used in the proof of Theorem 3.13, we realize that the fact that P and $\neg\neg P$ having different boundaries leads to the possibility that P is indeterminate but $\neg\neg P$ is false. This motivates us to posit the following definition of a proposition in another logic related to the topological structure of phase space:

Definition 4.1. For a set P_i in the topological space X, we define the *proposition* corresponding to P_i (denoted by P) to be the equivalence class $[P_i]$ defined by the equivalence relation of having the interior *and* the same boundary as P_i. That is, $P_j \sim P_i$ (and so $P_j \in [P_i] = P$) if and only if int P_j = int P_i and $\partial P_j = \partial P_i$.

The verification that the condition "same interior and same boundary as" gives an equivalence relation is straightforward. Unfortunately, this definition does not allow for definitions of connectives in terms of set operations as is the case for other derived logics for classical systems which have been previously described. This situation is remedied by the following two definitions and result; these provide a useful characterization of almost all (this restriction will be made clear by Definition 4.3) of the equivalence classes:

Definition 4.2. Let S be a set in the topological space X. Then a point y on the boundary of S is a *recreant point* if and only if every neighborhood of y contains a point of the interior of S^c. The set of all the recreant points of S is denoted by *rctS*.

Remark. There exist boundary points of some set which are neither adherent nor recreant points of the set. For example, if S is an everywhere dense subset of X with empty interior, then every point in X is in ∂S, but S has no adherent and no recreant points.

Definition 4.3. A proposition P is said to be *proper* if and only if $\emptyset \neq P_0 \neq X$ (where P_0 is the canonical representative of P).

Remark. While mathematically the restriction to proper propositions may appear less than desirable, physically speaking it is quite reasonable. After all, physical measurements verifying statements about completely empty subsets of a phase space or for statements about the entire phase space are often impossible to accomplish.

Theorem 4.4. For *nonempty* sets S_i, S_j, the condition $[\mathrm{int}(S_i) \cup rct(S_i)] = [\mathrm{int}(S_j) \cup rct(S_j)]$ is equivalent to the condition that $\mathrm{int}(S_i) = \mathrm{int}(S_j)$ and $\partial S_i = \partial S_j$.

Remark. One should avoid the temptation of thinking that the equivalence condition is equivalent to $[\mathrm{int}(S_i) \cup \partial(S_i)] = [\mathrm{int}(S_j) \cup \partial(S_j)]$. The following is a counterexample to this: Let $S_i = \mathrm{R}^2 = X$ and let S_j be the set of rational lattice points of R^2. We see that S_j is a dense subset of X with empty interior. Thus, in this case,

$$\begin{aligned}[\mathrm{int}(S_i) \cup \partial(S_i)] &= X \cup \emptyset \\ &= X \\ &= \emptyset \cup X \\ &= [\mathrm{int}(S_j) \cup \partial(S_j)]\end{aligned}$$

We also see that $\mathrm{int}(S_i) \neq \mathrm{int}(S_j)$.

Proof. Suppose that $[\mathrm{int}(S_i) \cup rct(S_i)] = [\mathrm{int}(S_j) \cup rct(S_j)]$ and let $x \in \mathrm{int}(S_i)$. We wish to show that $x \in \mathrm{int}(S_j)$. As the interior of a set and its boundary are disjoint sets, it is the case that either $x \in \mathrm{int}(S_j)$ or $x \in rct(S_j)$, but not both. Assume that $x \in rct(S_j)$; the contradiction at which we shall arrive will prove that $x \in \mathrm{int}(S_j)$.

By definition, there exist neighborhoods N_{xi} of x such that $N_{xi} \subseteq \mathrm{int}\, S_i$ and N_{xj} of x such that $N_{xj} \cap \mathrm{int}\, S_j^c \neq \emptyset$. As $x \in N_{xi} \cap N_{xj}$ we know that $N_{xi} \cap N_{xj} \neq \emptyset$. Also, since $N_{xi} \cap N_{xj}$ is a neighborhood of x and x is a recreant point we know that $(N_{xi} \cap N_{xj}) \cap \mathrm{int}\, S_j^c \neq \emptyset$. As every set appearing in $(N_{xi} \cap N_{xj}) \cap \mathrm{int}\, S_j^c$ is open, we have that the set $U = (N_{xi} \cap N_{xj}) \cap \mathrm{int}\, S_j^c$ is an open set. We note that $U \subseteq \mathrm{int}(S_i)$ and that $U \subseteq \mathrm{int}(S_j^c)$. Thus, every point of U is in $\mathrm{int}(S_i)$ and so is not in $rct(S_i)$. Also, every point of U is in $\mathrm{int}(S_j^c)$ and so is not in $\mathrm{int}(S_j)$ and is not in $\partial(S_j)$, in particular, no point of U in $rct(S_j)$. Thus, there is some point $u \in U$ such that $u \in (\mathrm{int}(S_i) \cup rct(S_i))$ and $u \notin (\mathrm{int}(S_j) \cup rct(S_j))$. This contradiction of our assumption proves that $x \in \mathrm{int}(S_j)$. As x is an arbitrary element of $\mathrm{int}(S_i)$, we have shown that $\mathrm{int}(S_i) \subseteq \mathrm{int}(S_j)$. A symmetric argument proves that $\mathrm{int}(S_j) \subseteq \mathrm{int}(S_i)$. Thus we have that $\mathrm{int}(S_i) = \mathrm{int}(S_j)$.

As the interior of a set and its boundary are disjoint, this result also yields the fact that $[\mathrm{int}(S_i) \cup rct(S_i)] = [\mathrm{int}(S_j) \cup rct(S_j)]$ implies that $rct(S_i) = rct(S_j)$. Our result together with Theorem 3.5 also implies that $adh(S_i) = adh(S_j)$). Thus, in order to prove that $\partial S_i = \partial S_j$ we must show that the boundary points of each set which are neither adherent nor recreant are the same. By the definitions of adherent and recreant points, the only points in ∂S which are not in $adh(S)$ nor in $rct(S)$ will occur in open subsets V of X

where $S \cap V$ is everywhere dense in V but such that $V \cap S$ has empty interior. In such cases $(V \cap S) \subseteq \partial(V \cap S)$. We also see that if $\mathrm{int}(S_i) = \mathrm{int}(S_j)$, $adh(S_i) = adh(S_j)$, and $rct(S_i) = rct(S_j)$, then those open subsets of X where S_i is everywhere dense with empty interior must agree with the corresponding sets for S_j. Thus, $[\mathrm{int}(S_i) \cup rct(S_i)] = [\mathrm{int}(S_j) \cap rct(S_j)]$ also implies that $\partial S_i = \partial S_j$.

The proof of the converse $\mathrm{int}(S_i) = \mathrm{int}(S_j)$ and $\partial S_i = \partial S_j$, implies that $[\mathrm{int}(S_i) \cap rct(S_i)] = [\mathrm{int}(S_j) \cap rct(S_j)]$ can now be proved using theorem 3.5. The proof also involves the fact concerning points in a boundary that are neither adherent nor recreant, discussed above. ∎

As the equivalence classes in this logic are not canonically represented by the interior of a set, we will not refer to it as an *open* phase space logic; instead, we will refer to it as "modified phase space logic." Using the characterization of our proposition provided by Theorem 4.3, the connectives in modified phase space logic are defined as follows:

Definition 4.5. Let P and Q be proper propositions. That is, P is the equivalence class $[P_i]$, and Q is the equivalence class $[Q_j]$. Then we define the logical operators $\neg$ ("not"), $\wedge$ ("and"), and $\vee$ ("or") as follows:

$$P \wedge Q = [(\mathrm{int}(P_i) \cup rct(P_i)) \cap (\mathrm{int}(Q_j) \cup rct(Q_j))] \tag{19}$$

$$P \vee Q = [(\mathrm{int}(P_i) \cup rct(P_i)) \cup (\mathrm{int}(Q_j) \cup rct(Q_j))] \tag{20}$$

$$\neg P = [(P_i^c)] = [(\mathrm{int}\, P_i^c) \cup rct(P_i^c)] \tag{21}$$

That the operations $\wedge$ and $\vee$ are well defined for proper propositions follows immediately from the distributivity of intersection over union and the associativity of the union operator. That the negation operator is well defined requires a bit more effort.

Theorem 4.6. The negation operator as defined in Definition 4.5 is well defined for all proper propositions.

Proof. Let P_i and P_j be any two elements of the proposition P. By definition we have that $\mathrm{int}(P_i) \cup rct(P_i) = \mathrm{int}(P_j) \cup rct(P_j)$; we wish to show that $\mathrm{int}(P_i^c) \cup rct(P_i^c) = \mathrm{int}(P_j^c) \cup rct(P_j^c)$. By Theorem 4.4 we know that $\mathrm{int}(P_i) = \mathrm{int}(P_j)$ and that $\partial(P_i) = \partial(P_j)$, so we have

$$\begin{aligned}\overline{P_i} &= \mathrm{int}(P_i) \cup \partial(P_i)\\ &= \mathrm{int}(P_j) \cup \partial(P_j)\\ &= \overline{P_j}\end{aligned}$$

By Theorem 3.3 we have that $(\overline{\mathrm{int}(P_i^c)})^c = \overline{(P_i^c)^c} = \overline{P_i}$. Hence, $(\mathrm{int}(P_i^c)) = ((\mathrm{int}(P_i^c))^c)^c = (\overline{P_i})^c$. Thus, as $\overline{P_i} = \overline{P_j}$, we have

$$(\mathrm{int}(P_i^c)) = (\overline{P_i})^c = (\overline{P_j})^c = (\mathrm{int}(P_j^c))$$

(with the final equality following from another application of Theorem 3.3). By the definition of the boundary of a set, we have $\partial(P_i^c) = \partial(P_i) = \partial(P_j) = \partial(P_j^c)$. Thus, as we have shown that $\mathrm{int}(P_i^c) = \mathrm{int}(P_j^c)$ and $\partial(P_i^c) = \partial(P_j^c)$, another application of Theorem 4.3 gives us the desired equality. ∎

The extension of the definitions of the connectives to nonproper propositions is straightforward but requires us to rely on Definition 4.1 to show that they are well defined, as the characterization given by Theorem 4.4 holds only for nonempty sets. Also one must distinguish between the empty proposition and the proposition whose canonical representative is the empty set $[\emptyset]$. The latter object consists of those nonempty sets with no recreant points and no interior. We must similarly distinguish between statements concerning the entire phase space X and the proper proposition $[X]$. Thus in modified phase space logic the empty proposition will be denoted by ω and the "entire" proposition will be denoted by α; of course ω will always be assigned the value false, while α will always be assigned the value true. With these comments in mind, the definitions of the connectives are extended as follows:

$$\alpha \wedge P = P \tag{22}$$

$$\alpha \vee P = \alpha \tag{23}$$

$$\omega \wedge P = \omega \tag{24}$$

$$\omega \vee P = P \tag{25}$$

$$\neg\alpha = \omega \tag{26}$$

$$\neg\omega = \alpha \tag{27}$$

Now that we have shown that the negation operator is well defined in modified phase space logic, we now exhibit the property which is the main difference between this logic and open phase space logic:

Theorem 4.7. In modified phase space logic, $\neg\neg P = P$.

Proof. Let $P_i \in P$; by the definition of the negation operator we have

$$\begin{aligned}\neg\neg P &= \neg[(\overline{P_i^c})]\\ &= [(\overline{(\overline{P_i^c})^c})]\\ &= [P_i]\\ &= P \quad \blacksquare\end{aligned}$$

In spite of the equality displayed in Theorem 4.6, this logic is not Boolean. In particular, *tertium non datur* does not hold. If we take P to be a proposition such that $P_i \in P$ is an everywhere dense set with empty interior then

$$\begin{aligned} P \vee \neg P &= [P_i] \vee \neg[P_i] \\ &= [P_i] \vee [(P_i)^c] \\ &= [\mathrm{int}(P_i) \cup rct(P_i)] \vee [\mathrm{int}(P_i^c) \cup rct(P_i^c)] \\ &= [\{\mathrm{int}(P_i) \cup rct(P_i)\} \cup \{\mathrm{int}(P_i^c) \cup rct(P_i^c)\}] \\ &= [\{\emptyset \cup \emptyset\} \cup \{\emptyset \cup \emptyset\}] \\ &= [\emptyset] \quad \blacksquare \end{aligned}$$

Remarks. 1. Recall the distinction between the proposition ω and $[\emptyset]$. It can be shown that $[\emptyset]$ is the only proposition which is always indeterminate; i.e., every possible measurement will assign a value of indeterminate to this proposition.

2. This same example demonstrates that the "law of noncontradiction" also fails to hold in this logic, in contrast to the situation in the three-valued open phase space logic. Indeed, if we simply substitute $\wedge$ for $\vee$ in the derivation above and use the appropriate definitions, we see that this is the case.

3. This example also shows that in modified phase space logic there is no analog to the weak version of *tertium non datur* found in three-valued open phase space logic (Theorem 3.10).

We now describe the assignment of truth values in this logic as follows (recall that we say that a measurement m verifies a proposition if and only if $m \subseteq P_i$, where $P_i \in P$):

Definition 4.8. Let P be a proposition in open phase space logic and let m be a measurement result. Suppose that P_i is any set in $[P]$. Then the proposition P will be assigned the truth value:

(1) **true** if m verifies P;
(2) **false** if m verifies $\neg P$;
(3) **indeterminate** if neither P nor $\neg P$ is verified.

In contrast with the three-valued open phase space logic described in Section 3, the negation operator is truth functional in the three-valued modified phase space logic described here.

Theorem 4.9. In three-valued modified phase space logic, if a measurement m:

(1) assigns a value of true to P, then m assigns a value of false to $\neg P$;
(2) assigns a value of false to P, then m assigns a value of true to $\neg P$;
(3) assigns a value of indeterminate to P, then m assigns a value of indeterminate to $\neg P$.

Proof. Let the measurement m verify the proposition P; that is, for any $P_i \in P$ it is the case that $m \subseteq P_i$. We wish to show that m verifies $\neg(\neg P)$. This follows immediately from Theorem 4.6. Thus m assigns a value of false to $\neg P$. The proof of case 2 follows immediately from Definition 4.7.

Let us now consider the third case: m assigns a value of indeterminate to P. We know that m cannot assign a value of true to $\neg P$, as item 1 implies that $\neg(\neg P)$ is assigned a value of false by m. As we know, $P = \neg(\neg P)$, so we have a contradiction. A similar contradiction is reached if we assume that m assigns a value of false to $\neg P$. Thus m must assign a value of indeterminate to $\neg P$. ■

At the beginning of this section we stated that we would discuss the status of modified phase space logic as a derived logic, the issue to which we now turn. Westmoreland and Schumacher (1993) described a derived logic as the consequence of "a physically motivated theory of measurement, which is in turn built upon the mathematical structure of a state space." The theory of measurement which was used to derive the three-valued open phase space logic described in Section 2 is described in Westmoreland and Schumacher (1993). Loosely speaking, this measurement theory allows for measurements of arbitrarily high but not infinite precision. We also recall that measurements are open subsets of phase space. The question of the whether open and closed phase space logics are derived logics is equivalent to whether or not distinct equivalence classes can be distinguished by open subsets of the phase space. Similarly, the question of modified phase space logic being a derived logic is equivalent to the question of whether or not equivalence classes in this logic can be distinguished by open sets.

That is to say: Can we distinguish sets with distinct interiors *or* distinct boundaries by open sets? Let us now consider the first possibility: the interiors of A and B are distinct. We recall from Section I, Definition 2 of Westmoreland and Schumacher (1993) that two sets, say A and B, are distinguishable by open sets if there is some open set U such that $U \subset A$ but $U \not\subset B$ or vice versa. If the interiors of sets C and D have distinct interiors, then int $C \not\subset$ int D or vice versa; without loss of generality, we may assume the former. In that case int C is the open set which distinguishes the equivalence classes $[C]$ and $[D]$.

We now turn our attention to the other possibility: the boundaries of A and B are distinct. We also recall from Section I, Definition 1 of Westmoreland and Schumacher (1993) that two sets, say E and F, are distinguishable by open sets if there is some open set U such that $U \cap E = \emptyset$ but $U \cap F \neq \emptyset$ or vice versa. Thus, if the boundaries of the sets A and B are distinct, then there is some point y such that $y \in \partial A$ but $y \notin \partial B$ or vice versa; without loss of generality, we may assume the former. In that case, for every open set U_y such that $y \in U_y$ it must be that $U_y \cap A \neq \emptyset$ and there is some open set V_y such that $y \in V_y$ and $V_y \cap B = \emptyset$. Thus, in this case we also have that A and B are distinguishable by open sets. We have now demonstrated that modified phase space logic is a derived logic.

5. TWO EXAMPLES REVISITED

Westmoreland and Schumacher (1993) applied bivalent closed phase space logic to the analysis of two elementary examples from the theory of classical dynamical systems. We will now analyze these examples using the two three-valued phase space logics developed here. We will find that in many respects the results of this analysis logic are intuitive and realistic.

Our first example consists of a particle of unit mass moving in one dimension (denoted by coordinate q) in a double potential well, as sketched in Fig. 1. In addition to the conservative force, the particle is also subject to a drag force that opposes the particle's velocity. There exist three equilibrium points in this system: the origin $q = 0$ and the points A and B. A particle initially at rest at one of these points will remain at that point.

The phase space of this system has two coordinates, which we may take to be q and $\dot{q}$. There are three fixed points in phase space, located at the equilibrium points along the q axis. The origin is a saddle point; but the points A and B are both attractors, i.e., states initially in sufficiently small neighborhoods of A or B approach these points asymptotically as $t \to \infty$.

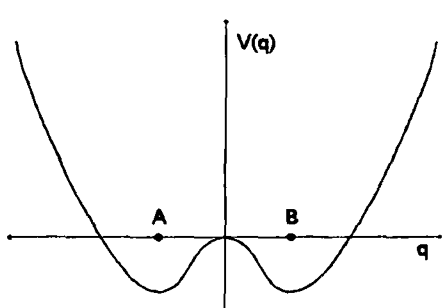

Fig. 1.

We can consider, for instance, the set of all points in the phase space which eventually approach A under the equations of motion. This is known as the *basis of attraction* of the attractor A, and is shown as the shaded region in Fig. 2. The unshaded region in Fig. 2 is the basin of attraction of B. The boundary between these two regions, which consists of points that approach neither A nor B, is called the *separatrix* of the two basins. (The separatrix consists of a pair of trajectories that asymptotically approach the saddle point at the origin as $t \to \infty$, one from each side.)

Consider the following statements:

- P_A: "The trajectory beginning at this point approaches A asymptotically."
- P_B: "The trajectory beginning at this point approaches B asymptotically."

These statements correspond to subsets of the phase space on which they are true. Naively, we can identify each statement with the basin of attraction of the corresponding point.

However, suppose we consider these as propositions within open phase space logic. Then the proposition P_A is an equivalence class of sets including those whose interiors are the basin of attraction of A. (Many other sets are also in this equivalence class; the smallest of them is the basin of attraction.) Similarly, P_B corresponds to sets including those whose interiors are the basin of attraction of B. This is because only those points which are in either basin of attraction can be verified by some measurement as a state through which a particle will pass on its way to A or B.

This has some interesting consequences. For instance, the propositions P_A and P_B are just the negations of one another: $P_A = \neg P_B$ and $P_B = \neg P_A$. The disjunction $P_A \vee P_B$ is no longer identically true, since the interiors of the basins do not cover the entire space. In fact, $P_A \vee P_B$ just corresponds to

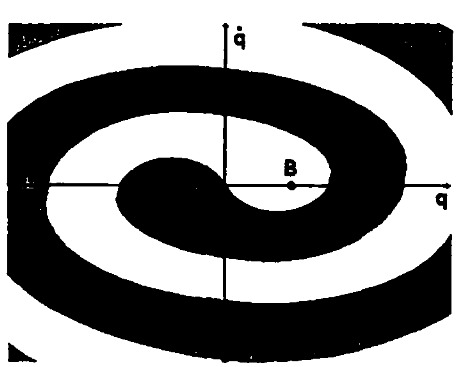

Fig. 2.

the complement of the separatrix. Finally, we note that the conjunction of P_A and P_B is identically false: $P_A \wedge P_B = 0$.

These observations concerning P_A and P_B are true regardless of whether we use a bivalent or a three-valued interpretation of open phase logic. The three-valued interpretation leads to statements which cannot be phrased in the bivalent version. For example, Theorem 2.10 implies that $P_A \vee P_B$ is never false; i.e., there is no measurement which will verify a state on the separatrix. Realistically, we know that the probability that the state of the system is *exactly* on the separatrix is zero. Thus, we would in practice conclude that the system must be in one basin or another (that is, $P_A \vee P_B$ must be true or unverified), and that it is in the basin for A if and only if it is not in the basin for B (that is, $P_A = \neg P_B$).

The fact that $P_A \vee P_B \neq [X]$ reflects the fact that points very near to (but not on) the separatrix are difficult to distinguish from it. (In fact, it may take a very long time for such a state to move away from the separatrix and approach either A or B.) If we employ a finite set of observations to determine in which basin of attraction the state of the system is, there will always be one or more possible outcomes—those outcomes that correspond to open sets containing points of the separatrix—that are consistent with either possibility. Any finite program of measurements used to determine the future destiny of the system will in some cases give rise to an ambiguous answer.

The analysis of this system under the modified phase space logic is very similar. This arises from the fact that neither basin of attraction possesses any recreant points. Thus, the equivalence classes of P_A and P_B consist of those sets which are equivalent to the basin of attraction of A (denoted by β_A) and the basin of attraction of B (denoted by β_B), respectively. We also have in this logic that $\neg P_A = P_B$ and vice versa. As we have that $[\beta_A \cup \beta_B] \neq [X] = 1$, we again have that *tertium non datur* does not hold in this case. It is true in this case, however, that the weaker version of *tertium non datur* does hold. As we noted earlier, this is not always the case in modified phase space logic.

Our second example is a more abstract dynamical system whose phase space is a torus. A state is described by two coordinates, q and p, each of which ranges over values between 0 and 1. Opposite sides of the resulting square are "pasted" together, forming a toroidal phase space. The dynamics of the system are given by

$$\frac{d}{dt} q = \alpha$$

$$\frac{d}{dt} p = \beta$$

where α and β are constants. The trajectory passing through a point (q_0, p_0) is this phase space (the *orbit* of this point) is shown in Fig. 3.

The structure of the orbits in this phase space is determined by the ratio β/α. If this ratio is a rational number, then the orbit will return to itself after a finite number of trips "around" the phase space. On the other hand, if the ratio is irrational, then the orbit will never return to itself. In fact, the orbit will be a dense subset of the phase space.

Consider the statement P: "The orbit of (q_0, p_0) contains this point." Naively, this statement corresponds to the orbit in the phase space. If we interpret this statement as a proposition in bivalent open phase space logic, we see this situation from a different perspective. If the ratio β/α is rational, then P corresponds to the orbit as before. However, if the ratio β/α is irrational, then the orbit, while dense, has empty interior. In either case, this proposition is "false" with respect to bivalent open phase space logic. This is because no set of observations can guarantee that the state of the system lies on a particular orbit; the orbit, though dense, has no interior. If we try to ensure that the system is on the orbit, we cannot succeed.

If we consider the proposition using the three-valued open phase space logic, the same result is obtained: P is always assigned the value false. In this case, this result follows from the fact that the elements of P have no interior and therefore no adherent points.

Alternatively, if we interpret this statement in the modified phase space logic, we obtain yet another perspective. If β/α is rational, then the orbit is not dense; thus, the complement has nonempty interior. Hence there are measurements of P (those open sets which intersect the orbit) under which P is indeterminate. But it is also the case that there exist measurements of P (those open sets which do not intersect the orbit) under which P is false. In contrast, if β/α is irrational, P is always indeterminate, as there are no measurements which will verify P nor the negation of P.

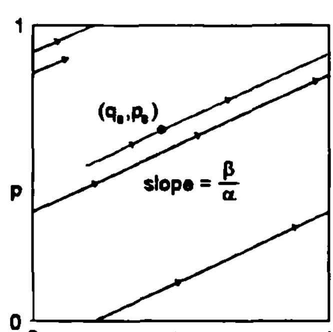

Fig. 3.

6. DISCUSSION

Perhaps the first issue we should address in this section is why we have presented two different three-valued logics in this paper. One reason is that the development of the two logics follows naturally in the way presented. The three-valued open phase space logic is suggested by the fact that *tertium non datur* does not hold in open phase space logic. Modified phase space logic, in turn, is motivated by the fact that the negation operator, even in three-valued open phase space logic, is not truth functional. It is the desire to find a logic in which the negation operator is truth functional which leads us to consider the modified phase space logic.

Another reason for the presentation of the two logics is that the differences and similarities between them are interesting and instructive. For example, the three-valued open phase space logic is developed by making changes in the bivalent phase space logic which relate solely to the *interpretation* of propositions under measurements. That is, the identity of propositions as mathematical structures is the same in the bivalent and three-value versions of open phase space logic. The interpretation which we assign to these propositions is different in the two logics: In the bivalent case we interpret propositions under a given measurement as being either verified or not verified. In the three-valued case we interpret propositions under a given measurement as being true, false, or indeterminate.

In contrast, in developing the modified phase space logic we are required to effect changes in the syntactic structural features of the logic, as well as in the interpretation of the propositions. That is, the identity of the propositions as mathematical structures is changed in passing from the open phase space logic to the modified phase space logic. The differences which arise between the two logics, given the differences in their development, is interesting.

The final reason which we shall give for the presentation of the two logics is that the way that each views propositions is quite different. The three-valued open phase space logic is firmly rooted in the verifiability approach to propositions concerning physical systems (Westmoreland and Schumacher, 1993). On the other hand, the modified phase space logic mixes the verifiability and falsifiability (Westmoreland and Schumacher, 1993) approaches to such propositions. This is reflected in the characterization of the equivalence classes provided by Theorem 4.6: the inclusion of the interior of a representative is related to the proposition's verifiability, while the inclusion of the set of recreant points is related to its falsifiability.

We now turn to a comparison between the three-valued logics developed here and the three-valued logic developed by Hans Reichenbach for the analysis of quantum mechanical systems. At first blush one is perhaps more cognizant of a major difference between the approaches: Reichenbach's system was developed for quantum mechanical systems, while the systems developed here are applicable to classical mechanical systems. While this distinction should be kept in mind, the fact that they are three-valued logics designed for the analysis of physical systems is important enough to make the closer examination of their similarities and differences worthwhile.

One of the most striking features of Reichenbach's system of three-valued logic is that it does not possess one negation operation, but three; they are defined as follows. If the proposition P has the values T, I, F (corresponding to "true," "indeterminate," and "false," respectively) the "complete negation" of P, $\bar{P}$, has the values I, T, and T respectively. (Note that we are here using Reichenbach's notation. The overbar in this instance has no connection with closure, the operation denoted by the overbar elsewhere in this paper.) The "cyclical negation," $\sim P$, is defined as having the values I, F, and T, respectively. Finally, the "diametrical negation," $-P$, is defined as having the values F, I, and T, respectively.

This feature of multiple negations is not possessed by either the three-valued open phase space logic nor the modified phase space logic. As the negation in the three-valued open phase space logic is not truth functional, it does not correspond exactly with any of the negations in Reichenbach's system. In contrast, by Theorem 4.9, the negation in the modified phase space logic corresponds exactly with the "diametrical negation" of Reichenbach's system.

Several writers (Hempel, 1945; Turquette, 1945; Nagel, 1944) have criticized Reichenbach's system for not explaining the exact meaning of the truth values T, F, and I. Indeed, Turquette (1945) notes that the criterion for I is described by terms such as "unknowable to Laplace's superman" or "unknowable in principle." No such criticism applies to either of the three-valued logics described here. The truth values are determined solely by the mathematical structures with which the propositions are identified and by the measurements run to verify them. This is a feature of derived logics in general: such logics reflect the fact that the verification of an empirical proposition about a physical system depends upon some process of measurement. As measurement is itself a physical process, the types of propositions which can be verified is restricted by the physical facts relating to the system.

Reichenbach was mindful of such considerations when he developed his system for quantum mechanics. For example, he makes the following statement (Reichenbach, 1944):

> The quantum mechanical significance of the truth-value *indeterminate* is made clear by the following consideration. Imagine a general physical situation s, in which we make a measurement of the entity q; in doing so we have once and forever renounced knowing what would have resulted if we had made a measurement of the entity p. It is useless to make a measurement of p in the new situation, since we know that the measurement of q has changed the situation.

Thus, while Reichenbach developed his system with the problem of measurement in mind, his system had no mechanism for assigning truth values based on measurements. This, of course, is precisely the problem identified by Hempel, Turquette, Nagel, and others.

This is in contrast with the approach taken by von Neumann and Birkhoff (1936) in the system which they developed for the analysis of quantum systems. In the von Neumann–Birkhoff logic, propositions are identified with closed subspaces of the Hilbert space which serves as the state space for the quantum mechanical system. In a Hilbert space H, closed subspaces may be identified with the projection operator which projects any vector in H onto that subspace. As not all projection operators on a Hilbert space commute, there are propositions, P and Q say, such that P and Q cannot both be measured for by any suite of measurements. As the von Neumann–Birkhoff logic is derived from the Hilbert space structure of the state space, it reflects this fact in a direct way.

Similarly, the three-valued open phase space logic and the modified phase space logic are derived from the topological state space of the classical systems. This provides a direct mechanism for assigning truth values which in no way depends upon epistemically vague mechanisms such as "Laplace's superman." We see that the logics presented here in some sense represent a synthesis of the approaches of Reichenbach and von Neumann–Birkhoff: they are three-valued systems, but the structure of the propositions is derived from the state space. Of course this is only in a weak sense, as the logics developed here are not designed for the analysis of quantum mechanical systems. This does suggest the intriguing possibility that such a synthesis may exist for quantum logics.

Geach (1972, pp. 195–198) makes an observation concerning three-valued logics to which we should respond here. The point Geach makes is that one must be careful about interpretations of the middle truth value, which he denotes by X. He points out that interpretations of X as "doubtful" or "as probable as not" are problematic. He states that, in the systems of three-valued logic he considers, the law of noncontradiction is not a tautology. Given that P has value X, it follows that $\neg P$ also has the value X, and so $P \wedge \neg P$ has the value X—not "false," as required by noncontradiction. This raises the question of whether or not Geach's objection applies to the systems presented here.

In the three-valued open phase space logic, the law of noncontradiction continues to be a tautology; i.e., for any proposition P it is the case that $P \wedge \neg P$ is assigned the value false by any measurement. This follows from the definition of the operator $\wedge$ in the open phase space logic:

$$P \wedge Q = [P_i \cap Q_j] \tag{28}$$

So, if we replace Q with $\neg P$ we have

$$\begin{aligned} P \wedge \neg P &= [P_i \cap P_i^c] \\ &= [\emptyset] \end{aligned}$$

Recall that in the three-valued open phase space logic $[\emptyset]$ is always false. Consequently, Geach's comment does not apply directly to this particular logic.

This is not the case, however, with the modified phase space logic. In this logic, we observed in the remarks following Theorem 4.7 that the law of noncontradiction does not hold in this logic, precisely the situation with which Geach was concerned. However, the law of noncontradiction fails in an interesting way: the statement $P \wedge \neg P$ is assigned the value of indeterminate by every possible measurement only when P, and so $\neg P$, is the equivalence class of sets which are everywhere dense in the phase space but have empty interior. (It is a straightforward exercise to show that this collection forms an equivalence class.) In general, $P \wedge \neg P$ can be assigned the value indeterminate by a measurement m only if P_0 has some recreant points—points in regions of phase space where P and $\neg P$ are indistinguishable by measurement.

In more natural language, this corresponds to: the measurement m does not warrant the truth of either P nor $\neg P$. Again the question is: Does Geach's criticism apply to this interpretation as it does to interpretations such as "P is doubtful" or "P is as likely as not"? Let us consider the always-indeterminate proposition; i.e., the equivalence class of sets which are everywhere dense with empty interior. Such propositions do arise in physical systems; recall that the example in Section 5, whose phase space is a torus, involves such a proposition when the ratio of the parameters is irrational. For such a proposition, it is indeed the case that both P and $\neg P$ are assigned the value of indeterminate (in fact $P = \neg P$ as equivalence classes in this system), just as Geach observes. Is it invalid to then say that $P \wedge \neg P$ is also assigned the value of indeterminate by any measurement?

Given the interpretation used here, it is not invalid to make such a statement. One must be careful not to confuse propositions with subsets of the phase space: propositions are equivalence classes of subsets of the phase space. Measurements will not be able to distinguish between subsets, but only between classes of subsets. In our example, the fact that $P \wedge \neg P$ is always indeterminate reflects the fact that no measurement will verify either $P \wedge \neg P$ or its negation (which is also always indeterminate—and thus equivalent to $P \wedge \neg P$ and P and $\neg P$!). Indeed, a closer analysis of the logic in this case implies that no representative P_0 of P can be distinguished from its set complement P_0^c by any measurement of finite precision. This result, however surprising, does reflect the theory of measurement in an informative way.

Thus we see that the interpretation of the modified phase space logic is not invalidated by Geach's criticism. First of all, the failure of noncontradiction only occurs when a proposition and its negation are inseparably intertwined in some region of phase space. Ordinary examples [such as Geach's (1972) "It is probable that it will and will not rain tomorrow"] will rarely exhibit this property. In those cases in which this mixing of P and $\neg P$ does occur (such as the second example in Section 5), the ambiguity of measurement results yields a more subtle notion of indeterminacy than merely being "doubtful" or "as probable as not." This is perhaps easier to appreciate at the level of logical structure than at the level of natural language.

The developments presented here raise several interesting questions. One, which we noted previously, is the possibility of a three-valued derived logic for the analysis of quantum mechanical systems. There seem to be at least two possible approaches to developing such a system. One is to modify the theory of measurement so that the quantum mechanical facts of life are reflected in the axioms for measurement. The second approach would be to look for substructures of the quantum mechanical state space which can provide a natural three-valued interpretation.

Another question which arises from this development is how the three-valued logics developed here compare to the three-valued logics developed by others, such as Łukasiewicz (1970) and Rosser and Turquette (1952). Our three-valued logics appear as very natural developments in the program of measurement-based "derived logics" for physical systems. We have only touched on the relation of these systems of logic to that of Reichenbach. Further comparisons should deepen our understanding of these and other three-valued systems.

REFERENCES

Birkhoff, G., and von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823–843.

Geach, P. T. (1972). *Logic Matters*, University of California Press, Berkeley.

Gibbins, P. (1987). *Particles and Paradoxes: The Limits of Quantum Logic*, Cambridge University Press, Cambridge.

Hempel, C. G. (1945). *Journal of Symbolic Logic*, **10**, 97–100 [cited in Jammer (1974)].

Jammer, M. (1974). *The Philosophy of Quantum Mechanics*, Wiley, New York.

Lewis, C. I., and Langford, C. H. (1932). *Symbolic Logic*, Century Company, New York [cited in Rosser and Turquette (1952)].

Łukasiewcz, J. (1970). *Selected Works*, L. Borkowski, ed., North-Holland, Amsterdam [cited in Jammer (1974)].

Munkres, J. F. (1974). *Topology: A First Course*, Prentice-Hall, Englewood Cliffs, New Jersey.

Nagel, E. (1944). Logic without ontology, in *Naturalism and the Human Spirit*, Y. H. Krikorian, ed., Columbia University Press, New York, pp. 210–241 [reprinted in H. Feigl and W. S. Sellars, eds., *Readings in Philosophical Analysis*, Appleton, New York (1949), pp. 191–210].

Reichenbach, H. (1944). *Philosophic Foundations of Quantum Mechanics*, University of California Press, Berkeley [Sections 29–37 reprinted in C. A. Hooker, ed., *The Logico-Algebraic Approach to Quantum Mechanics*, Reidel, Dordrecht, Holland (1975)].
Rosser, J. B. (1941). *American Journal of Physics*, **9**, 207–212 [cited in Rosser and Turquette (1952)].
Rosser, J. B., and Turquette, A. R. (1952). *Many-Valued Logics*, North-Holland, Amsterdam.
Turquette, A. R. (1945). *Philosophic Review*, **54**, 513–516 [cited in Jammer (1974)].
Vickers, S. (1989). *Topology Via Logic*, Cambridge University Press, Cambridge.
Westmoreland, M. D., and Schumacher, B. W. (1993). *Physical Review A*, **48**, 977–985.

Erratum

Three-Valued Derived Logics for Classical Phase Spaces

Michael D. Westmoreland, Benjamin W. Schumacher, and Steven C. Bailey

The above paper from *International Journal of Theoretical Physics*, **35**, 31–62 (1996) contains an erroneous definition. Definition 4.3 (page 47) should be replaced by the following:

A proposition P is said to be *proper* if and only if there is no open set U such that P_0 is dense in U but $\text{int}(P_0 \cap U) = \emptyset$, where P_0 is the canonical representative of P.

Theorem 4.4 should then be restricted to sets which are elements of proper propositions.

The motivation for this corrected definition is the same as that for the original definition: the measurement status of everywhere dense sets is problematic. For example, in the 2 dimensional Cartesian coordinate plane, there is no finite collection of finite precision measurements which will distinguish between the set of points with rational x-coordinate and the set of points with irrational x-coordinate. It was originally thought that these problems could be avoided by excluding the proposition where the canonical representative was the entire space and, by duality, excluding the empty proposition. We have since realized that this reasoning is incorrect. In a sense, the measurement problem related to everywhere dense, empty interior sets is scale invariant. That is, in order to have distinguishable propositions, we must exclude those which are locally dense with empty interior; hence, the corrected definition.

It should be noted that there is a three-valued derived logic for phase spaces which, in general, is not equivalent to the ones described in the paper but it does avoid the notion of proper and improper propositions. The description of this logic is beyond the scope of the subject paper and certainly beyond the scope of this correction.